Homework 6 Solution

1. Sec. 2.5 Q14

14. Prove the converse of Exercise 8: If A and B are each $m \times n$ matrices with entries from a field F, and if there exist invertible $m \times m$ and $n \times n$ matrices P and Q, respectively, such that $B = P^{-1}AQ$, then there exist an n-dimensional vector space V and an m-dimensional vector space W (both over F), ordered bases β and β' for V and γ and γ' for W, and a linear transformation T: V $\to$ W such that

$$A = [\mathsf{T}]_\beta^\gamma \quad \text{and} \quad B = [\mathsf{T}]_{\beta'}^{\gamma'}.$$

Hints: Let $\mathsf{V} = \mathsf{F}^n$, $\mathsf{W} = \mathsf{F}^m$, $\mathsf{T} = \mathsf{L}_A$, and β and γ be the standard ordered bases for F^n and F^m, respectively. Now apply the results of Exercise 13 to obtain ordered bases β' and γ' from β and γ via Q and P, respectively.

Let $V = F^n$, $W = F^m$. Let $\beta = \{e_1 \cdots e_n\}$, $\gamma = \{\varepsilon_1 \cdots \varepsilon_m\}$ be the standard ordered basis for V and W resp.

Let $\beta' = \{Qe_1, \cdots, Qe_n\}$ $\quad \gamma' = \{P\varepsilon_1, \cdots, P\varepsilon_m\}$

Since Q is invertible and β is a basis, β' is also a basis for V. Similarly, γ' is also a basis for W

$$[I_V]_{\beta'}^{\beta} = Q \qquad [I_W]_{\gamma'}^{\gamma} = P$$

Let $T = L_A : V \longrightarrow W$

$\qquad x \longmapsto Ax$

Then $[T]_\beta^\gamma = A$

$$[T]_{\beta'}^{\gamma'} = [I_W]_\gamma^{\gamma'} [T]_\beta^\gamma [I_V]_{\beta'}^{\beta} = P^{-1} A Q = B$$

2. Sec. 2.6 Q7(b)

7. Let $V = P_1(R)$ and $W = R^2$ with respective standard ordered bases β and γ. Define $T: V \to W$ by

$$T(p(x)) = (p(0) - 2p(1), p(0) + p'(0)),$$

where $p'(x)$ is the derivative of $p(x)$.

(a) For $f \in W^*$ defined by $f(a, b) = a - 2b$, compute $T^t(f)$.
(b) Compute $[T^t]_{\gamma^*}^{\beta^*}$ without appealing to Theorem 2.25.
(c) Compute $[T]_\beta^\gamma$ and its transpose, and compare your results with (b).

(b) $\beta = \{1, x\}$ $\qquad \gamma = \{e_1, e_2\}$

Let $\beta^* = \{f_1, f_2\}$ $\qquad \gamma^* = \{g_1, g_2\}$

$\begin{cases} f_1(1) = 1 \\ f_1(x) = 0 \end{cases} \Rightarrow f_1(a+bx) = a f_1(1) + b f_1(x) = a$

$\begin{cases} f_2(1) = 0 \\ f_2(x) = 1 \end{cases} \Rightarrow f_2(a+bx) = a f_2(1) + b f_2(x) = b$

$\begin{cases} g_1(e_1) = 1 \\ g_1(e_2) = 0 \end{cases} \Rightarrow g_1((a,b)) = a \cdot g_1(e_1) + b \cdot g_1(e_2) = a$

$\begin{cases} g_2(e_1) = 0 \\ g_2(e_2) = 1 \end{cases} \Rightarrow g_2((a,b)) = a \cdot g_2(e_1) + b \cdot g_2(e_2) = b$

$$\begin{aligned} T^t(g_1)(a+bx) &= g_1 \circ T(a+bx) \\ &= g_1((-a-2b, a+b)) \\ &= -a-2b \\ &= -f_1(a+bx) - 2 \cdot f_2(a+bx) \qquad \forall a+bx \in P_1(R) \end{aligned}$$

$$[T^t(g_1)]_{\beta^*} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$T^*(g_2)(a+bx) = g_2 \circ T(a+bx)$$
$$= g_2((-a-2b, a+b))$$
$$= a+b$$
$$= f_1(a+bx) + f_2(a+bx) \quad \forall\, a+bx \in P_1(R)$$

$$[T^*(g_2)]_{\beta^*} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore $\quad [T^*]_{\gamma^*}^{\beta^*} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$

3. Sec. 2.6 Q10(abc)

10. Let $\mathsf{V} = \mathsf{P}_n(F)$, and let $c_0, c_1, \ldots, c_n$ be distinct scalars in F.

(a) For $0 \leq i \leq n$, define $\mathsf{f}_i \in \mathsf{V}^*$ by $\mathsf{f}_i(p(x)) = p(c_i)$. Prove that $\{\mathsf{f}_0, \mathsf{f}_1, \ldots, \mathsf{f}_n\}$ is a basis for V^*. *Hint:* Apply any linear combination of this set that equals the zero transformation to $p(x) = (x - c_1)(x - c_2) \cdots (x - c_n)$, and deduce that the first coefficient is zero.

(b) Use the corollary to Theorem 2.26 and (a) to show that there exist unique polynomials $p_0(x), p_1(x), \ldots, p_n(x)$ such that $p_i(c_j) = \delta_{ij}$ for $0 \leq i \leq n$. These polynomials are the Lagrange polynomials defined in Section 1.6.

(c) For any scalars $a_0, a_1, \ldots, a_n$ (not necessarily distinct), deduce that there exists a unique polynomial $q(x)$ of degree at most n such that $q(c_i) = a_i$ for $0 \leq i \leq n$. In fact,

$$q(x) = \sum_{i=0}^{n} a_i p_i(x).$$

(a) consider $a_0 f_0 + \cdots + a_n f_n = 0$.

Let $p_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} (x - c_j)$, $\begin{cases} p_i(c_j) = 0 \text{ if } j \neq i \\ p_i(c_i) \neq 0 \end{cases}$

Then $(a_0 f_0 + \cdots + a_n f_n)(p_i) = 0$

$\Rightarrow a_0 f_0(p_i) + \cdots + a_n \cdot f_n(p_i) = 0$

$\Rightarrow a_0 \cdot p_i(c_0) + \cdots + a_n \cdot p_i(c_n) = 0$

$\Rightarrow a_i\, p_i(c_i) = 0$

$\Rightarrow a_i = 0$

Therefore $\{f_0 \cdots f_n\}$ is L.I. in V.

Since $|\{f_0, \cdots, f_n\}| = n+1 = \dim(V)$

we know that $\{f_0 \cdots f_n\}$ is a basis for V.

(b) • By corollary, there exists a dual basis $\{\hat{p}_0, \cdots, \hat{p}_n\} \subset V^*$ of $\{f_0 \cdots f_n\}$. i.e. $\hat{p}_i(f_j) = \delta_{ij}$.

$$\delta_{ij} = \hat{p}_i(f_j) = f_j(p_i) = p_i(c_j)$$

• And The Lagrange polynomial satisfies this eqn.

$$p_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} (x - c_j) \Big/ \prod_{\substack{j=0 \\ j \neq i}}^{n} (c_i - c_j)$$

satisfies $p_i(c_j) = \delta_{ij}$

• if $p_i(c_j) = \delta_{ij}$ and $q_i(c_j) = \delta_{ij}$

Then $(p_i - q_i)(c_j) = 0 \quad \forall j$

Then $p_i - q_i = 0$, $p_i = q_i$

So such $\{p_i\}$ is unique.

(c) since $\{p_0 \cdots p_n\}$ is a basis for $P_n(F)$

$\forall q \in P_n(F)$ is uniquely represented as

$$q(x) = \sum_{i=0}^{n} a_i \cdot p_i(x)$$

$$q(c_i) = \sum_{j=0}^{n} a_j p_j(c_i) = \sum_{j=0}^{n} a_j \delta_{ij} = a_i$$

4. Sec. 5.1 Q13

13. Let T be a linear operator on a finite-dimensional vector space V over a field F, let β be an ordered basis for V, and let $A = [\mathsf{T}]_\beta$. In reference to Figure 5.1, prove the following.

(a) If $v \in \mathsf{V}$ and $\phi_\beta(v)$ is an eigenvector of A corresponding to the eigenvalue λ, then v is an eigenvector of T corresponding to λ.

(b) If λ is an eigenvalue of A (and hence of T), then a vector $y \in \mathsf{F}^n$ is an eigenvector of A corresponding to λ if and only if $\phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

(a) if $A \cdot \phi_\beta(v) = \lambda \cdot \phi_\beta(v)$

Then $[T(v)]_\beta = [T]_\beta \cdot [v]_\beta = [T]_\beta \cdot \phi_\beta(v) = A \cdot \phi_\beta(v)$

$= \lambda \cdot \phi_\beta(v) = \phi_\beta(\lambda v) = [\lambda v]_\beta$

i.e $T(v) = \lambda v$

(b)

$Ay = \lambda y$

$\Leftrightarrow$ $[T]_\beta \cdot \phi_\beta(\phi_\beta^{-1}(y)) = \lambda \cdot \phi_\beta(\phi_\beta^{-1}(y))$

$\Leftrightarrow$ $[T]_\beta [\phi_\beta^{-1}(y)]_\beta = \lambda [\phi_\beta^{-1}(y)]_\beta$

$\Leftrightarrow$ $[T(\phi_\beta^{-1}(y))]_\beta = [\lambda \phi_\beta^{-1}(y)]_\beta$

$\Leftrightarrow$ $T(\phi_\beta^{-1}(y)) = \lambda \phi_\beta^{-1}(y)$

5. Sec. 5.1 Q23

23. Use Exercise 22 to prove that if $f(t)$ is the characteristic polynomial of a diagonalizable linear operator T, then $f(\mathsf{T}) = \mathsf{T}_0$, the zero operator. (In Section 5.4 we prove that this result does not depend on the diagonalizability of T.)

$f(t) = \det([T]_\beta - tI)$

T is diagonalizable, then $\exists$ basis $\beta = \{v_1 \cdots v_n\}$ such that $T(v_j) = \lambda_j v_j$

Then $f(\lambda_i) = 0 \quad \forall i = 1 \cdots n$

$f(T)(v_i) \overset{\text{by ex 22}}{=} f(\lambda_i) \cdot v_i = 0 \cdot v_i = \vec{0} \quad \forall i = 1 \cdots n$

Since β is a basis, $f(T) = T_0$ the zero transformation